AN ONTO-SEMIOTIC APPROACH TO REPRESENTATIONS IN MATHEMATICS EDUCATION

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Research in the didactics of mathematics has shown the importance that representations have in teaching and learning processes as well as the complexity of factors related to them. Particularly, one of the central open questions that the use of representations poses is the nature and diversity of objects that carry out the role of representation and of the objects represented. The objective of this article is to show how the notion of semiotic function and mathematics ontology, elaborated by the onto-semiotic approach to mathematics knowledge, enables us to face such a problem, by generalizing the notion of representation and by integrating different theoretical notions used to describe mathematics cognition.

Importance of representations in mathematics education

To speak about representation is equivalent to speaking about, for example, knowledge, meaning, comprehension and modelling. Without doubt, these notions make up one of the central nuclei, not only of our discipline, but also of epistemology, psychology and other sciences and technologies that study human cognition, its nature, origin and development. This diversity of disciplines interested in knowledge representation is the reason for the diversity of approaches and ways of conceiving it.

Current notions of representation show the different components and facets implied in mathematics activity (Goldin and Janvier, 1998; Goldin, 2002) and the situations in which language and personal and cultural objects arising from this activity are developed. In our opinion, the complexity and ambiguity of knowledge representations rest on the following: to talk about mathematics knowledge representation necessarily implies speaking about mathematics knowledge, hence about mathematics activity, its cultural and cognitive productions and also those related to the world that surrounds us.

In this article, we will look at the *ontological problem* of the representations and other related questions from the holistic approach which proposes the onto-semiotic approach to cognition and mathematics instruction (Godino and Batanero, 1998; Godino, 2002; Godino, Batanero and Roa, 2005; Contreras, Font, Luque and Ordóñez, 2005). The notion of semiotic function and the ontology that proposes this theoretical approach generalises and clarifies the notion of representation and provides a solution to the aforementioned ontological problem. To be more precise, we are going to deal with the following problematic aspects of the representations:

1. The nature of the objects that intervene in the representations.
2. The problem of the representation of the generic element.
3. The role that the representation of one object plays.

In this first part of the article, the problem and the objectives are posed. Then, in the next section, we briefly present the theoretical framework of the onto-semiotic approach showing the solution proposed to the ontological problem of representation and meaning. In the third part, we reflect on the role of *generic element* in mathematics and its relation to representations. In the fourth part, we study the problem of considering that there is one same mathematical object that has multiple different representations. And, finally, in the final part, we present a synthesis of the response given by the onto-semiotic approach to the questions posed, ending with some general conclusions.

In Figure 1, we present an episode from a class that we are going to use as a context of reflection, to illustrate the type of application that we do considering representations and using the theoretical constructs elaborated by the onto-semiotic approach. This example involves responses given by two secondary school students (17 years old) to an item of a worksheet proposed in the study process of the derivative.

The ontological problem of representations and meaning

The onto-semiotic approach to mathematics cognition tackles the problem of meaning and knowledge representation by elaborating an explicit mathematical ontology based on anthropological (Bloor, 1983; Chevallard, 1992), semiotic and socio-cultural theoretical frameworks (Ernest, 1998; Presmeg, 1998; Sfard, 2000; Radford, 2003; Radford, 2006). It assumes socio-epistemic relativity for mathematical knowledge since knowledge is considered to be indissolubly linked to the activity in which the subject is involved and is dependent on the cultural institution and the social context of which it forms part (Radford, 1997).

We now synthesize the ontology proposed in the onto-semiotic approach to mathematics cognition.

Systems of operative and discursive practices linked to fields or types of problems

All kinds of performances or expressions (*e.g.*, verbal and graphic), carried out by someone in order to solve mathematics problems, communicate the solution obtained to others, validate it or generalise it to other contexts and prob-

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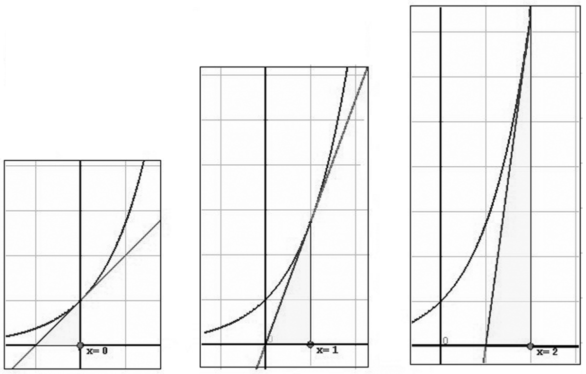
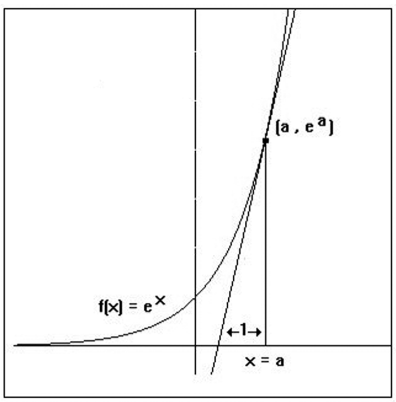
Worksheet

In the computer classroom you have observed that the function $f(x) = e^x$ fulfils the fact that all the sub-tangents are of the same length, 1. Using this property:

a) Calculate $f'(0)$, $f'(1)$ and $f'(2)$



b) Calculate $f'(a)$



c) Prove that the derivative function of $f(x) = e^x$ is the function $f'(x) = e^x$.

Students' responses to section c:

VÍCTOR:

The derivative function of $f(x) = e^x$ is $f'(x) = e^x$ because the derivative of a function at one point is equal to the slope of the straight line tangent at this point.

The slope is achieved by dividing $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$, in this function $x_2 - x_1$ is always given by 1, and by dividing the vertical increment, which is the e^x, by the horizontal increment, which is 1, gives us e^x.

ROCIO:

$f'(x) = \frac{f(x)}{sub\tan gent}$; as $f(x) = e^x$

$f'(x) = \frac{e^x}{1} = e^x$

Figure 1: An episode from a class as a context for reflection: worksheet proposed to a group of students (17 years old); part of the process of study of the derivative and the students' two correct answers to section c.

lems, are considered to be mathematical practice (Godino and Batanero, 1998). These practices might be idiosyncratic (*e.g.*, the students' answers in Figure 1) or be shared within an institution (*e.g.*, the teacher's practices implemented in the mathematics class). An institution is constituted by the people involved in the same class of problem-situations, whose solution implies the carrying out of certain shared social practices and the common use of particular instruments and tools. Institutions are conceived as *communities of practices* and they include, for instance, school classes or ability groupings and ethnic groups. Mathematical practices are carried out by persons and institutions in the context of material, biological and cultural backgrounds. Therefore, we assume a socio-epistemic relativity for systems of practices, emergent objects and meanings.

In the study of mathematics, rather than a particular practice to solve a specific problem, it is interesting to consider the systems of practices (operative and discursive) carried out by people when faced with problematic types of situations. It is proposed to answer such questions as, what is the mathematical object 'arithmetical average'?, or, what does the expression 'arithmetic average' mean?, with:

> the system of practices that a person carries out (personal meaning) or when shared institutionally (institutional meaning) to solve a type of problem situation in which it is necessary to find a representative value of a set of data.

Intervening and emerging objects of the system of practices

In mathematical practices, ostensive objects (*e.g.*, symbols and graphs) and non-ostensive objects (which we bring to mind when doing mathematics) that are textually, orally, graphically or even gesturally represented, intervene. New objects that come from the system of practices and explain their organization and structure (types of problems, procedures, definitions, properties, arguments) emerge [1]. If the systems of practices are shared in the heart of an institution, the emerging objects are considered to be *institutional objects*, whilst if these systems correspond to a person we consider them as *personal objects*. In Figure 1, we can observe that the students share some practices as a result of the teaching (*e.g.*, they use the property that all the sub-tangents of the exponential function are equal to 1); but there are also differences in the practices of others (such as, use of graphical representations or not and different symbolism). From the system of practices carried out in the classroom a new object emerges: the derivative of $f(x) = e^x$ is $f'(x) = e^x$; the justification of this proposition is another emergent object, which is different to the proof given at university level, or even by different students.

Relations between objects: semiotic function

We use Hjelmslev's (1943) notion of *function of sign* (named by Eco (1979) as *semiotic function*), the dependence between a text and its components and between these components themselves. In the onto-semiotic approach a semiotic function is conceived, interpreting this idea, as the correspondences (relations of dependence or function) between an antecedent (expression, signifier) and a consequent (content, signified or meaning), established by a subject (person or institution) according to a certain criteria or corresponding code. These codes can be rules (habits, agreements) that inform the subjects about the terms that should be put in correspondence in the fixed circumstances.

For us, the relations of dependence between expression and content can be representational (one object which is put in place of another for a certain purpose), and instrumental (an object uses another or others' objects as an instrument). In this way, semiotic functions and the associated mathematics ontology take into account the essentially relational nature of mathematics and generalize the notion of representation: the role of representation is not totally undertaken by language (oral, written, graphical, gestures, …). In accordance with Peirce's semiotics, it is proposed that the different types of objects (problem situations, procedures, concepts, properties and arguments), can also be expressions or content of the semiotic functions.

In the example (see Figure 1), there is a network of representational semiotic functions: the exponential function is designated by graphic and algebraic symbolism; the concept of tangent, sub-tangent and derivate are also represented by words and symbols. But, as we will explain in the following two sections, the general notions of function and derivative are represented by the particular examples of the exponential function and its derivative, respectively. The graphical representation is also used as a tool to develop a 'proof' of the property that all the sub-tangents are equal to 1.

Configuration of objects

The notion of *systems of practices* is useful for certain types of macro-didactic analysis, particularly when comparing the specific way in which mathematical knowledge arises in different institutional frameworks, contexts of use or language games (Wittgenstein, 1953). For a more precise description of mathematics activity it is necessary to introduce six types of primary entities: situations, procedures, languages, concepts, properties and arguments. In each case, these objects will be related among themselves forming *configurations*, defined as the network of emerging and intervening objects of the systems of practices and the relations established between them. These configurations can be epistemic (networks of institutional objects) or cognitive (network of personal objects). The systems of practices and the configurations are proposed as theoretical tools to describe the mathematical knowledge, in its double personal and institutional version.

The six types of primary objects suggested here widen the traditional distinction between conceptual and procedural entities that we consider insufficient to describe the objects intervening and emerging from mathematical activity. The problem – situations promote and contextualise the activity; language (symbols, notations, graphics, …) represent other entities and serve as tools for action; arguments justify the procedures and properties that relate the concepts. These entities have to be considered as functional and relative to the language game (institutional frameworks and use contexts) in which they participate; they have also a recursive character, in the sense that each object might be composed of other entities. Depending on the analysis level for example arguments, these entities might involve, for example, concepts, properties and operations.

Cognitive dualities

The notion of language game (Wittgenstein, 1953) occupies an important place when considered together with the notion of institution: these are the contextual factors that relativize the meanings of the mathematical objects and attribute a functional nature to them. The mathematical objects that intervene in mathematical practices and those emerging from them, depending on the language game they are taking part in, can be considered from the following facets or dual dimensions: personal-institutional, unitary-systemic, expression-content, ostensive-non-ostensive and extensive-intensive (Godino, 2002). These facets are grouped in pairs that are dually and dialectically complemented. They are considered as attributes applicable to the different primary and secondary objects, giving rise to different 'versions' of the said objects. In Godino, Batanero and Roa (2005) the six types of primary entities and the five types of cognitive dualities are described using examples from research in the field of combinatoric reasoning.

The types of objects described, summarised in Figure 2 (systems of practices, emerging entities, configurations or onto-semiotic networks, the cognitive dualities or contextual attributes, together with the notion of semiotic function as the basic relational entity) make up an operative response to the ontological problem of representation and meaning of mathematical knowledge.

In the following sections we will show how the five dimensions or cognitive dualities, as well as the other theoretical instruments elaborated by the onto-semiotics approach and in particular the notion of semiotic function enable us to face the complexity that research on knowledge representation requires. Furthermore, we will try to relate these facets with different problematic aspects of the representations that other authors have dealt with.

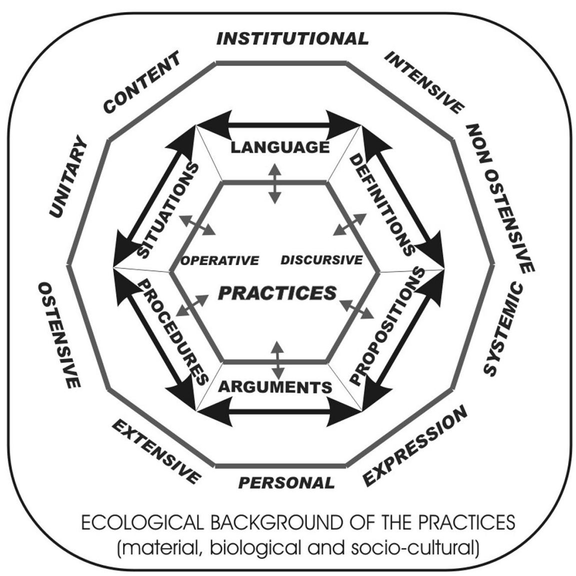


Figure 2: Onto-semiotics of mathematical knowledge.

The problem of representing generic elements

One of the crucial characteristics of mathematics activity is the use of generic elements, that is, a set or system of elements considered as one unit. This practice can be useful in the process of definition; for example, a rational number is a class of ordered pairs of integer numbers that satisfy a relation; the generic element $(a, b) \in Z \times (Z - \{0\})$ is none other than the scheme which includes many pairs of the same class, for example, [(1, 2), (5, 10), (3, 6), ...], thought in one act of thinking. At other times, the generic element is useful for an economy of thought: for example, the fact that the three heights of a triangle come together at the same point does not depend on the type of triangle we are talking about, so any attempt to demonstrate this should refer to a scheme of possible triangles and not one specific one.

However, a dialectic between the generic element and the general element can, frequently, cause a greater cognitive complexity to arise. The mathematics reasoning, to go from the general to general, introduces an intermediate phase that consists of contemplating an individual object. This fact poses a serious dilemma: if reasoning has to be applied to a specific object, it is necessary for there to be some guarantee that in doing so we reason about any object, so that it is possible to justify the generalisation in which reasoning ends. Furthermore, since the specific object is associated with its representation, the problem of whether the representation refers to a specific object or to a general concept appears (D'Amore, 2005).

The introduction of the extensive/intensive duality in the onto-semiotic approach can help to clarify the problem of the use of generic elements (Contreras, Font, Luque and Ordóñez, 2005). Two questions, which are different but connected, have to be considered with respect to this problem:

1. Why does an intermediate phase, which refers to a specific object, intervene in the demonstration of a mathematical proposition (the statement of a definition, etc.)?

2. How is it possible that, in spite of this, reasoning in which there is a similar intermediate phase gives rise to a universal conclusion?

The particular element normally forms part of a chain in which the previous links are generic elements. At the same time, the particular element, to be considered as generic, will be converted into the previous link of a new particular case and so on.

The extensive/intensive facet becomes an essential instrument to analyse the complexity associated with these three aspects. Expressed differently, the use of the generic element is associated with a complex net of semiotic functions (and so representations) that relate intensive with extensive objects. We will show this with the example of the students' responses included in Figure 1.

If we observe the three sections of the worksheet (Figure 1) we can see that, in the statement, the step from the particular to the general has been taken into account. In *question a*, students are asked to calculate the derivative for the three specific values (0, 1 and 2). In *question b*, they are asked to calculate the derivative for a specific value 'a' and, in *question c*, for any value. That is, the change from extensive to intensive is present in the design of the worksheet. In this process, we can observe that the extensive objects 'represent' the intensive ones.

In order to calculate the derivative function from a condition that satisfies all the tangents, the student has to identify the following net of semiotic functions:

1. Treat the variables related by the formula and the graph of the exponential base e, separately. To do this, it is necessary to understand the exponential function of base e as a process in which other objects, one being x and the other being $f(x)$, intervene. Here, a semiotic function that relates the object $f(x)$ to the object x, is established, having an instrumental role.

2. Associate x to the slope of the tangent line to the point on the x axis. This relation can be considered as a semiotic function that relates the object x with the object *slope of the tangent line to the point of the x axis*.

3. Associate the expression that permits us to calculate the slope of the tangent line to the point of the x axis with $f'(x)$. In this case, we have a semiotic function that relates one notation with another different but equivalent one.

4. Consider x as a variable. In this case, we have a semiotic function that relates an object to the class it belongs to.

5. Understand the function obtained as a particular case of the 'derivative function' class. In this case, we have a semiotic function that relates an object to the class it belongs to.

If we look at the worksheet handed to the students we can observe that the sequence of sections aims at making the establishment of these semiotic functions, easier. The use of the letter a and the equality $x = a$, in *question b* of the worksheet, have the role of introducing a specific element in the student's reasoning and so make step 1, easier. The reason for including the use the graph and the symbolic notation together is that the teacher wants the students to carry out steps 2 and 3. Steps 4 and 5 are intended to be achieved from *question c*.

This example permits us to illustrate a phenomena that we consider to be very relevant: the student, in order to carry out the majority of mathematical practices, has to activate a net of complex semiotic functions and the ostensive objects used are determinant, both to reduce or increase the complexity of this net or to carry out the practice correctly. For example, if we had eliminated *question b* in the worksheet, we would still want the student to apply the technique to calculate the derivative function and we would still use graphs (the ones from the previous activity with the computer and those belonging to *question a*) and symbolic expressions (*question c*). However, the complexity of the semiotic functions that the student would have to carry out would increase considerably and so also would the possibilities of solving the task.

When we use a representation in mathematical practices as a generic element we are acting on a specific object, but we situate ourselves in a 'language game' in which it is understood that we are interested in its general characteristics and we disregard the particular aspects. The analysis of dialogues between teachers and students related to the use of generic elements (for example those mentioned in Contreras, Font, Luque and Ordóñez, 2005), is necessary to know the details about the characteristics of this language game and of the difficulties that students have to take part in it. The knowing and understanding of the rules (or not) of this language game is fundamental to the make up of the net of semiotic functions associated with the practices in which the generic element intervenes.

The problem of multiple representations of the 'same' mathematical object

Frequently we say that one same mathematical object (such as, function or derivative) is given by certain representations (such as, algebraic, graphs or tables). [2] We think that this way of conceiving the role of the representations in mathematics and in the conceptualisation processes is a little naïve.

It is enough to look with a historic perspective at any mathematical object to illustrate the complexity of the relations that are established between a mathematical object, its associated ostensive and the situations in which the object is used (in addition to the ostensive and associated practices) to organise phenomena. Consider the cissoid, as an example, defined [3] as a geometrical locus in the framework of synthetic geometry. The definition of the cissoid enables us to represent it by the drawing of a curve. In fact, in the construction carried out by Cabri Géomètre software (see Figure 3), the cissoid is represented by tracing point P when moving point M.

If we situate ourselves in the framework of analytic geometry and we use analogous techniques to those used by Descartes in *The geometry* we can obtain the following *representation* of the cissoid: $x^3 + y^2 x - ay^2 = 0$. This translation 'Curve ⇒ symbolic equation' is a technique that does not live alone but needs a theoretical background that justifies the move and allows it to make sense.

The research programme, initiated by Descartes, is a global programme in which local study is not considered. While we limit ourselves to look for an implicit expression, we move from a global point of view. However, when we consider obtaining the explicit expression of the cissoid we are obliged to introduce local reasoning. When situated in this new context (local perspective), the development of techniques in power series enable us to obtain an explicit expression for the cissoid. A historical look also shows that the different ostensive forms that can represent a mathematical object are the result of long evolution, where, in some cases, a new form of representation gives rise to a new avenue for research.

For mathematics education, we consider that it is important to show the ingenuity of the point of view that considers the ostensive representations of mathematical objects simply to be different meanings of the same object. This consideration tends to underestimate the importance of the different ostensive representations, the configuration of the objects considered and the translations among them in the production of the global meaning of the said object [4] (Wilhelmi, Godino and Lacasta, 2005).

The fact that the ostensive representations are framed in research programmes and that they imply the use of configurations or complex onto-semiotic networks, has serious implications. Here are three of the most important:

1. Representations cannot be understood on their own. An equation or a specific formula, a particular graph in a Cartesian system only acquires meaning as part of a larger *system* with established meanings and conventions. So, it is more convenient to speak about epistemic configurations (or cognitive, if we refer to personal systems of practices) rather than to speak about ostensive representations or signs in order to make clear the net of objects and relations involved when the semiotic register or the context of use are changed.
2. As the same object can be classified in two different research programmes or historic-cultural settings, each one with their systems of representations, each representation can be converted into a 'represented object' of the representation of the other research program. When the cissoid is studied in the framework of analytic geometry, a complex net of semiotic functions, whose beginning and end can be represented by Figure 4, is activated:
3. So, depending on the context, the curve can provide a geometric representation of the equation, or

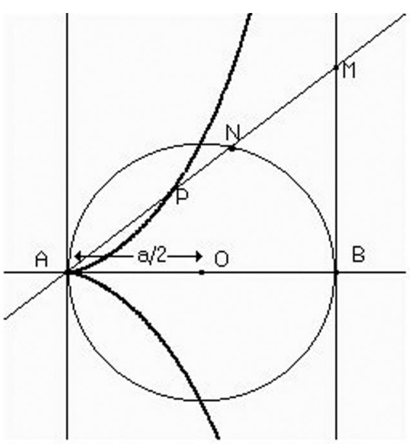


Figure 3: Tracing the cissoid.

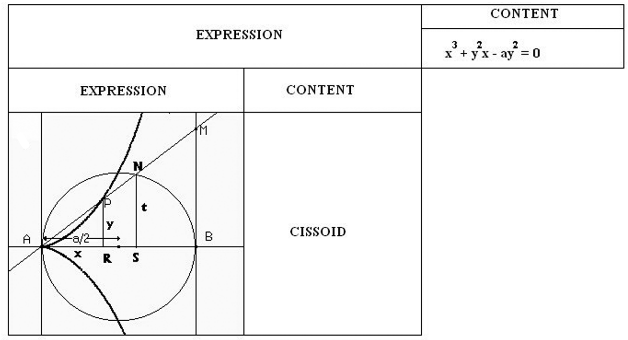


Figure 4: A complex net of semiotic functions for the cissoid.

the equation can provide an algebraic symbolism of the curve. This fact leads us to consider that the cissoid can be represented by a curve in *synthetic geometry* and by an equation in *analytic geometry*.

4. An ostensive representation, on the one hand has a representational value: it is something that can be put in place of something different to itself and on the other hand, it has an instrumental value: it permits specific practices to be carried out that, with another type of representation, would not be possible. The representational aspect leads us to understand representation in an unitary way as 'something' for 'something'. However, the instrumental value leads us to understand representation in a systemic way, like an 'iceberg' of a complex system of practices that the said representation makes possible.

In the onto-semiotic approach, the introduction of the unitary-systemic duality in the analysis of the representations enables us to reformulate the naïve vision that *there is one 'same' object with different representations*. What there is, is a complex system of practices in which each one of the different pairs object/representation (without segregating them) makes possible a subset of the set of practices that are considered to be the meaning of the object. Expressed differently, the object, considered as emergent from a system of practices, can be considered as unique and with a holistic meaning. However, in each subset of practices, the object/representation pair (without segregation) is different, in the sense that it makes different practices possible.

In the example of the worksheet (Figure 1), the use of graphic representation with dynamic software is necessary to find a condition that fulfils all the tangents (the starting point of the worksheet). In order to answer *question a* approximately, all you need is the graphic representation; however to answer it exactly, it is also necessary to use the symbolic expression of the exponential function. To answer *question b*, it is necessary to use both graphic and symbolic representations. The technique that the school institution intends the students to apply in this worksheet is only possible if the graphic and the symbolic representations are introduced at the same time. If the graphic representation is not contemplated, the technique is not viable. [5] Contemplating the graphic representation, in addition to the symbolic representation, enables us to carry out specific practices that, with the symbolic representation alone would not be possible.

Synthesis and conclusions

In this article, we have described some problematic aspects of the use of representations in mathematics education and we have given a response from the theoretical framework that we name the onto-semiotic approach.

With respect to the problem of representation of abstract entities, we propose analysing it in terms of the cognitive duality extensive-intensive. When we use an ostensive as a generic element in mathematical practices, we are acting on a particular object, but we situate ourselves in a 'language game' in which, when we refer to this particular object, it is understood that we are interested in its general characteristics and we disregard the particular aspects.

From the onto-semiotic point of view, the problem of whether there is one 'same' mathematical object that has different multiple representations, is naïve. The introduction of the unitary-systemic duality in the analysis of the representations permits the reformulation of this vision in the following way: What there is, is a complex system of practices in which each one of the different object/representation pairs (without segregation) permits a subset of practices of the set of practices that are considered as the meaning of the object.

In conclusion, we firstly want to point out that an onto-semiotic approach to representation and meaning is a holistic glance on these issues, which permits the great complexity associated with the use of these notions in mathematics education to be taken into account. This holistic glance helps to understand the phenomena of representation and meaning as the visible part of the 'complex iceberg' in the base of which we find ourselves with a net of objects, practices and associated ostensive objects, structured in epistemic (and cognitive) configurations.

Secondly, we point out that to understand representation in terms of semiotic function, as a relation between an expression and a content established by 'someone', has the advantage of not segregating the object from its representation. However, since this advantage is important, we wish to point out another that is even more so. We refer to the fact that in the onto-semiotic approach we propose that the expression and the content can be any type of object, filtered by the remaining dualities, which provides a greater analytic and explanatory capacity. Furthermore, the type of relations between expression and content can be varied, not only be representational, *e.g.*, "is associated with"; "is part of"; "is the cause of/reason for". This way of understanding the semiotic function enables us great flexibility, not to restrict ourselves to understanding 'representation' as being only an object (generally linguistic) that is in place of another, which is usually the way in which representation seems to us mainly to be understood in mathematics education.

Notes

[1] The notion of mathematical conceptual object is similar to that proposed in Radford's (2006) semiotic-cultural approach: "[…] mathematical objects are conceptual forms of historically, socially, and culturally embodied, reflective, mediated activity." (p. 59) However, in the *onto-semiotic approach* we propose a wider range of mathematical objects, which are not restricted to concepts.

[2] A central objective in mathematic teaching for some authors is making the students capable of changing from one representation to another: "The conversion of representations is a crucial problem when learning mathematics" (Duval, 2002, p. 318).

[3] Let C be a circumference with a radius a/2 and centre O, AB a diameter of C and l the straight line tangent to C at B. For each straight line AM, $M \in l$, we consider its intersection N with C and a segment AP, $P \in AM$, of the same length as MN. The geometric locus of the points P obtained is a curve called *Diocles's Cissoid*.

[4] In keeping with the anthropological stance, the *global meaning* is conceived, in the onto-semiotic approach, as the articulation of the partial subsystem of practices in which mathematical objects intervene in different institutions, contexts of use and language games.

[5] It is possible to calculate $f'(x)$ using only the symbolic expression of $f(x)$ from the limit definition of derivative.

[The references can be found on page 14 (ed.)]

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[These references follow on from page 7 of the article "An onto-semiotic approach to representations in mathematics education" that starts on page 2 (ed.)]

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